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NOTE ON THE CONVERGENCE OF DEFINITE INTEGRALS.

By J. K. Whittemore.

Let $f(x)$ be a function of x continuous for all values of x greater than a. Then a sufficient condition for the convergence of the improper definite integral,

$$\int_b^\infty f(x)\, dx \qquad b > a > 0$$

is

$$(1) \qquad \lim_{x=\infty} \left(x^k f(x)\right) = A \qquad k > 1$$

where A is any finite number. If this condition is satisfied, then

$$(2) \qquad \lim_{x=\infty} \left(x f(x)\right) = 0.$$

Equation (2) is, however, not a sufficient condition. For it is satisfied by $f(x) = \dfrac{1}{x \log x}$, and the integral

$$\int_2^\infty \frac{dx}{x \log x}$$

is divergent. Moreover equation (2) is not a necessary condition, for if $f(x) = \dfrac{\sin x}{x}$, the condition (2) is not satisfied; but the integral

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

Equation (2) is not a necessary condition for the convergence of the integral even when the integrand does not change sign. For the integral

$$\int_0^\infty \frac{dx}{1+x^4\sin^2 x}$$

is convergent,* and the expression, $\dfrac{x}{1+x^4\sin^2 x}$, approaches no limit as x is indefinitely increased. *Equation* (2) *is, however, a necessary condition for the convergence of the integral:*

$$\int_b^\infty f(x)\,dx$$

if $xf(x)$ has not an infinite number of maxima and minima for $x > b$. It is easy to see that in the two examples:

$$xf(x) = \sin x, \qquad xf(x) = \frac{x}{1+x^4\sin^2 x}$$

the function has an infinite number of maxima and minima beyond every value of x. The assumptions we make are the following:

1. $f(x)$ is continuous for $x > a$.
2. $\int_b^\infty f(x)\,dx$ is convergent, and
3. $xf(x)$ has not an infinite number of maxima and minima for $x > b > a$.

Then, we wish to prove that:

$$\lim_{x=\infty}\Big(xf(x)\Big) = 0.$$

A necessary and sufficient condition for the convergence of the integral,

$$\int_b^\infty f(x)\,dx$$

is

$$\lim_{\substack{a_1=\infty\\ a_2=\infty}}\left(\int_{a_1}^{a_2} f(x)\,dx\right) = 0 \qquad b < a_1 < a_2 \tag{3}$$

* See Appell — *Eléments d'Analyse mathématique*, p. 243.

From (3) it follows that to every positive ϵ there corresponds a quantity $d > b$ such that for $d < a_1 < a_2$,

$$\left| \int_{a_1}^{a_2} f(x)\, dx \right| < \epsilon.$$

Since $1/x$ does not change sign between $x = a_1$ and $x = a_2$ we may apply the mean value theorem for integrals. Then

$$\int_{a_1}^{a_2} f(x)\, dx = \int_{a_1}^{a_2} x\, f(x)\, \frac{dx}{x}$$

$$= \xi f(\xi) \int_{a_1}^{a_2} \frac{dx}{x} \qquad a_1 < \xi < a_2$$

$$= \xi f(\xi) \log \frac{a_2}{a_1}.$$

Let us choose $a_1 = 2d$, $a_2 = 2\, e\, d$, where e is the exponential base.

Then we have, as required, $d < a_1 < a_2$, and $\log \frac{a_2}{a_1} = 1$, and

$$\left| \int_{a_1}^{a_2} f(x)\, dx \right| = \left| \xi f(\xi) \right| < \epsilon. \tag{4}$$

We may now choose d so large that $xf(x)$ has neither maximum nor minimum for $x > d$, since by hypothesis there are not an infinite number of such points. Then $|x f(x)|$ must either never increase or never decrease as x increases indefinitely, and from (4) it follows that it must decrease indefinitely:

$$\lim_{x = \infty} \left(x f(x) \right) = 0.$$

In the same way we may see that a necessary condition for the convergence of the integral,

$$\int_a^b f(x)\, dx$$

where $f(x)$ is continuous for all values of x satisfying the inequalities,

$$a < x \leqq b,$$

and where $f(a) = \infty$, if $(x - a) f(x)$ has not an infinite number of maxima and minima in the neighborhood of $x = a$, is

$$\lim_{x = a} \left[(x - a) f(x) \right] = 0.$$

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